

On the intersection of homoclinic classes in intransitive sectional-Anosov flows

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Abstract

We show that if X is a *Venice mask* (i.e. nontransitive sectional-Anosov flow with dense periodic orbits, [9], [25], [24],[18]) supported on a compact 3-manifold, then the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit. In addition, we prove that the intersection of two different homoclinic classes in the maximal invariant set of a sectional-Anosov flow can be decomposed as the disjoint union of, singular points, a non-singular hyperbolic set, and regular points whose *alpha-limit set* and *omega-limit set* is formed by singular points or hyperbolic sets.

1 Introduction

The dynamical systems theory is interested to describes the behavior as time goes to infinity for the majority of orbits in a determinated system. An important tool for hyperbolic sets is the known *connecting lemma* [15], [2], [10]. Specifically, the lemma says that if X is an Anosov flow on a compact manifold M and $p, q \in M$ satisfy that for all $\varepsilon > 0$ there is a trajectory from a point ε -close to p to a point ε -close to q , then there is a point $x \in M$ such that $\alpha_X(x) = \alpha_X(p)$ and $\omega_X(x) = \omega_X(q)$.

In [7] was proved a similar result for sectional-Anosov flows, which is known as *sectional-connecting lemma*. Recall, the sectional hyperbolic sets and sectional Anosov flows were introduced in [21] and [19] respectively as a generalization of the hyperbolic sets and Anosov flows to include important examples such as the saddle-type hyperbolic attracting sets, the geometric and multidimensional Lorenz attractors [1], [11], [14] and certain robustly transitive sets. A fundamental hypothesis in the sectional-hyperbolic case consists in the alpha-limit set of $p \in$

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$M(X)$ to be non-singular. As the unstable manifold of every singularity σ of a sectional-Anosov X is contained in the maximal invariant set $M(X)$, would be interesting to know what is the omega-limit set of a point in $W_X^u(\sigma)$. In fact, it can be seen as a extension of the *sectional-connecting lemma*.

On the other hand, the class of Venice masks (i.e. intransitive sectional-Anosov flows with dense periodic orbits) has a particular interest since its existence shows that the spectral decomposition theorem [29] is not valid in the sectional-hyperbolic case. Its study has been collected by different authors during the last years. The examples exhibited in [9], [18], [25] are characterized because the maximal invariant set can be decomposed as the disjoint finite union of homoclinic classes. In addition, the intersection between two different homoclinic classes is contained in the closure of the union of the unstable manifold of the singularities. Specifically, this intersection can be decomposed as the disjoint union of, a singularity σ , a closed orbit C , and regular points such that its *alpha-limit set* is σ and the *omega-limit set* is C . Particularly, was proved in [25], [24] that every Venice mask with a unique singularity has these properties.

In search of properties which allow to characterized the dynamic of Venice masks, will be studied the behavior of homoclinic classes and its relation with the unstable manifolds of the singularities.

Let us state our results in a more precise way.

Consider a Riemannian compact manifold M of dimension n (a *compact n -manifold* for short). M is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$. We denote by ∂M the boundary of M . Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denotes by X_t the flow of X , $t \in \mathbb{R}$.

The ω -*limit set* of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \rightarrow \infty} X_{t_n}(p)$ for some sequence $t_n \rightarrow \infty$. The α -*limit set* of $p \in M$ is the set $\alpha_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \rightarrow \infty} X_{t_n}(p)$ for some sequence $t_n \rightarrow -\infty$. The *non-wandering set* of X is the set $\Omega(X)$ of points $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Given $\Lambda \in M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if $\Lambda = \cap_{t > 0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.

The *maximal invariant set* of X is defined by $M(X) = \bigcap_{t \geq 0} X_t(M)$.

Definition 1.1. A compact invariant set Λ of X is hyperbolic if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that

- E^X is the vector field's direction over Λ .
- E^s is contracting, i.e., $\|DX_t(x)|_{E_x^s}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- E^u is expanding, i.e., $\|DX_{-t}(x)|_{E_x^u}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

A compact invariant set Λ has a *dominated splitting* with respect to the tangent flow if there are an invariant splitting $T_\Lambda M = E \oplus F$ and positive numbers K, λ such that

$$\|DX_t(x)e_x\| \cdot \|f_x\| \leq Ke^{-\lambda t} \|DX_t(x)f_x\| \cdot \|e_x\|, \quad \forall x \in \Lambda, t \geq 0, (e_x, f_x) \in E_x \times F_x.$$

Notice that this definition allows every compact invariant set Λ to have a dominated splitting with respect to the tangent flow (See [8]): Just take $E_x = T_x M$ and $F_x = 0$, for every $x \in \Lambda$ (or $E_x = 0$ and $F_x = T_x M$ for every $x \in \Lambda$). A compact invariant set Λ is *partially hyperbolic* if it has a *partially hyperbolic splitting*, i.e., a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow whose dominated subbundle E is contracting in the sense of Definition 1.1.

The Riemannian metric $\langle \cdot, \cdot \rangle$ of M induces a 2-Riemannian metric [27],

$$\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_p M.$$

This in turns induces a 2-norm [13] (or areal metric [17]) defined by

$$\|u, v\| = \sqrt{\langle u, u/v \rangle_p} \quad \forall p \in M, \forall u, v \in T_p M.$$

Geometrically, $\|u, v\|$ represents the area of the parallelogram generated by u and v in $T_p M$.

If a compact invariant set Λ has a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow, then we say that its central subbundle F is *sectionally expanding* if

$$\|DX_t(x)u, DX_t(x)v\| \geq K^{-1}e^{\lambda t} \|u, v\|, \quad \forall x \in \Lambda, u, v \in F_x, t \geq 0.$$

By a *sectional-hyperbolic splitting* for X over Λ we mean a partially hyperbolic splitting $T_\Lambda M = E \oplus F$ whose central subbundle F is sectionally expanding.

Definition 1.2. A compact invariant set Λ is sectional-hyperbolic for X if its singularities are hyperbolic and if there is a sectional-hyperbolic splitting for X over Λ .

Definition 1.3. We say that X is a sectional-Anosov flow if $M(X)$ is a sectional-hyperbolic set.

The Invariant Manifold Theorem [3] asserts that if x belongs to a hyperbolic set H of X , then the sets

$$W_X^{ss}(p) = \{x \in M : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow \infty\} \quad \text{and} \\ W_X^{uu}(p) = \{x \in M : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow -\infty\},$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E_p^s and E_p^u of $T_p M$ respectively.

$$W_X^s(p) = \bigcup_{t \in \mathbb{R}} W_X^{ss}(X_t(p)) \quad W_X^u(p) = \bigcup_{t \in \mathbb{R}} W_X^{uu}(X_t(p))$$

are also C^1 immersed submanifolds tangent to $E_p^s \oplus E_p^X$ and $E_p^X \oplus E_p^u$ at p respectively.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

Definition 1.4. We say that a singularity σ of a sectional-Anosov flow X is Lorenz-like if it has three real eigenvalues $\lambda^{ss}, \lambda^s, \lambda^u$ with $\lambda^{ss} < \lambda^s < 0 < -\lambda^s < \lambda^u$. such that the real part of the remainder eigenvalues are outside the compact interval $[\lambda^s, \lambda^u]$. $W_X^s(\sigma)$ is the manifold associated to the eigenvalues with negative real part. The strong stable foliation associated to σ and denoted by $\mathcal{F}_X^{ss}(\sigma)$, is the foliation contained in $W_X^s(\sigma)$ which is tangent to space generated by the eigenvalues with real part less than λ^s .

Definition 1.5. A periodic orbit of X is the orbit of some p for which there is a minimal $t > 0$ (called the period) such that $X_t(p) = p$. An orbit is called closed if it is a periodic orbit or a singularity.

A homoclinic orbit of a hyperbolic periodic orbit O is an orbit $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_q M = T_q W^s(O) + T_q W^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that γ is a transverse homoclinic orbit of O . The homoclinic class $H(O)$ of a hyperbolic periodic orbit O is the closure of the union of the transverse homoclinic orbits of O . We say that a set Λ is a homoclinic class if $\Lambda = H(O)$ for some hyperbolic periodic orbit O .

Definition 1.6. A Venice mask is a sectional-Anosov flow with dense periodic orbits which is not transitive.

If A is a compact invariant set of X we denote $Sing_X(A)$ the set of singularities of X in A , and $Sing(X) = Sing_X(M(X))$. The closure of $B \subset M$ is denoted by $Cl(B)$. With these definitions we can state our main results.

2 Main statements

We show that if X is a *Venice mask* supported on a compact 3-manifold, then the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit. In addition, we prove that the intersection of two different homoclinic classes in the maximal invariant set of a sectional-Anosov flow can be decomposed as the disjoint union of, singular points, a non-singular hyperbolic set, and regular points whose *alpha-limit set* and *omega-limit set* is formed by singular points or hyperbolic sets.

Specifically, we have the following statements.

Theorem A. *If X is a three-dimensional Venice mask and σ is a singularity of X , then for every $q \in W_X^u(\sigma)$ such that q is non-recurrent we have the following dichotomy:*

- $\omega_X(q) \in \text{Sing}(X)$.
- $\omega_X(q) = O$, where O is a hyperbolic periodic orbit.

Theorem B. *The intersection of two different homoclinic classes H_1, H_2 in the maximal invariant set of a sectional-Anosov flow X is the disjoint union of a set S (possibly empty) of singularities, a non-singular hyperbolic set H (possibly empty), and a set R (possibly empty) of regular points such that if $q \in R$ then $\alpha_X(q) \subset H \cup S$ and $\omega_X(q) \subset H \cup S$.*

3 Preliminaries

We mention the following results which are essentials to proving the theorems.

Theorem 3.1 ([26]). *Let Λ be a sectional-hyperbolic set with dense periodic orbits. Then, every $\sigma \in \text{Sing}_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap \mathcal{F}_X^{ss}(\sigma) = \{\sigma\}$.*

We observe that $W_X^s(\sigma) \setminus \mathcal{F}_X^{ss}(\sigma)$ is decomposed by two connected components $W_X^{s,+}(\sigma)$ and $W_X^{s,-}(\sigma)$ (see figure 3). Hence for a Venice mask, a regular point in $M(X)$ contained in the stable manifold of some singularity σ , necessarily is contained either $W_X^{s,+}(\sigma)$ or $W_X^{s,-}(\sigma)$.

Lemma 3.2 (Hyperbolic lemma [26]). *A compact invariant set without singularities of a sectional-hyperbolic set is hyperbolic saddle-type.*

Remark 3.3. *Theorem 3.1 and the Hyperbolic Lemma imply that every Venice mask has singularities, and these are Lorenz-like.*

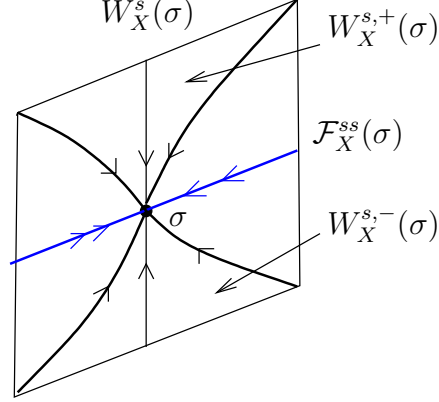


Figure 1: Connected components.

Definition 3.4. We say that a C^1 vector field X with hyperbolic closed orbits has the Property (P) if for every periodic orbit O there is a singularity σ such that

$$W_X^u(O) \cap W_X^s(\sigma) \neq \emptyset. \quad (1)$$

The above definition is useful by the interesting fact below.

Lemma 3.5. Every point in the closure of the periodic orbits of a vector field with the Property (P) is accumulated by points for which the omega-limit set is a singularity.

Moreover, we have an important property.

Lemma 3.6 ([25]). Every sectional-Anosov flow with singularities and dense periodic orbits on a compact 3-manifold has the Property (P).

Remark 3.7. By Lemma 3.5 and Lemma 3.6 we can assert that every Venice mask X has the Property (P) and $W^s(\text{Sing}(X)) \cap M(X)$ is dense in $M(X)$.

Definition 3.8. Given $\Sigma \subset M$ we say that $q \in M$ satisfies Property $(P)_\Sigma$ if $Cl(O^+(q)) \cap \Sigma = \emptyset$ and there is open arc I in M with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for every $x \in I$.

We finish to exhibit the preliminar statements with the following characterization.

Theorem 3.9 ([6]). Let X be a C^1 vector field in a compact 3-manifold M . If $q \in M$ has sectional-hyperbolic omega-limit set $\omega(q)$, then the following properties are equivalent:

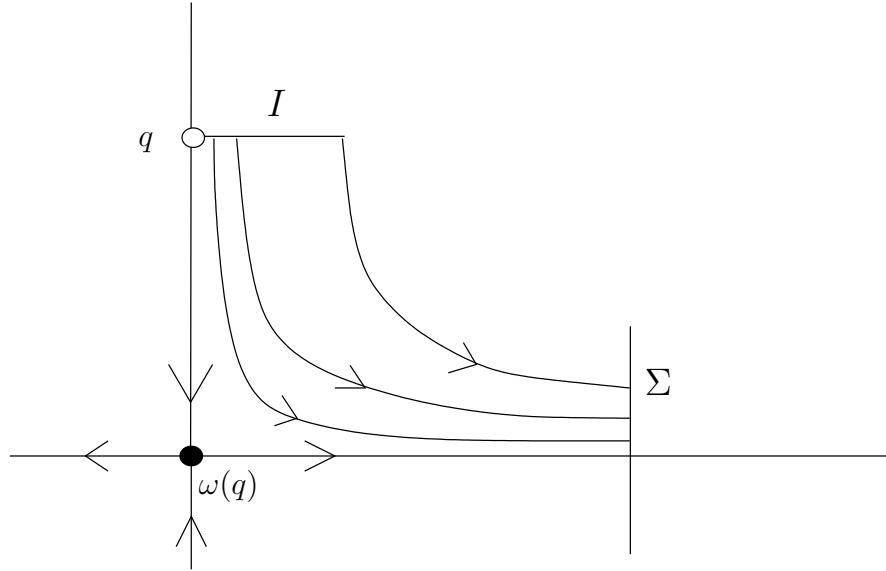


Figure 2: Property $(P)_\Sigma$.

- $\omega(q)$ is a closed orbit.
- q satisfies $(P)_\Sigma$ for some closed subset Σ .

In Figure 2 is exhibited the case when the omega-limit set $\omega(q)$ of the point q is a hyperbolic singularity of saddle-type.

4 Characterizing the omega-limit set

In this section we will prove the *Theorem A*. The idea is to consider a sequence of points satisfying the Property $(P)_\Sigma$, which approximates a point q in the unstable manifold of a fixed singularity. We show that q satisfies the Property $(P)_\Sigma$ too. Hereafter in this section, we assume that every regular point $q \in W^u(\text{Sing}(X))$ is non-recurrent.

First, we mention some facts of topology. Given a compact metric space (Y, d) , define a distance function between any point x of Y and any non-empty set B of Y by:

$$d(x, B) = \inf\{d(x, y) | y \in B\}.$$

Now, consider the collection $\mathcal{C}(Y) = \{C \in Y : C \text{ is a non-empty compact subset of } (Y, d)\}$. For $\mathcal{C}(Y)$, take the Hausdorff

metric d_H defined as the distance function between any two non-empty sets A and B of Y by:

$$d_H(A, B) = \sup\{d(x, B) | x \in A\}.$$

Lemma 4.1. *Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of closed sets contained in a compact metric space (Y, d) , such that $A_n \rightarrow A$ in the Hausdorff metric induced by d . Then $\partial A_n \rightarrow \partial A$.*

For now and on this section, let M be a riemaniann compact 3-manifold, and let X be a Venice mask on M . So, for a hyperbolic point p of X , $W_X^s(p)$ is just denoted by $W^s(p)$. The same interchanging s by u .

4.1 Existence of singular partitions

We introduce the following definition which can also be found in [4] and [5], and extends the notion given in [23].

A cross section of X is a codimension one submanifold S transverse to X . We denote the interior and the boundary (in topological sense) of S by $Int(S)$ and ∂S respectively. If $\mathcal{R} = \{S_1, \dots, S_k\}$ is a collection of cross sections we still denote by \mathcal{R} the union of its elements. Moreover

$$\partial \mathcal{R} := \bigcup_{i=1}^k \partial S_i \quad \text{and} \quad Int(\mathcal{R}) := \bigcup_{i=1}^k Int(S_i)$$

The size of \mathcal{R} will be the sum of the diameters of its elements.

Definition 4.2. *A singular partition of an invariant set H of a vector field X is a finite disjoint collection \mathcal{R} of cross sections of X such that $H \cap \partial \mathcal{R} = \emptyset$ and*

$$H \cap Sing(X) = \{y \in H : X_t(y) \notin \mathcal{R}, \forall t \in \mathbb{R}\}.$$

For a Lorenz-like singularity σ , the center unstable manifold $W_X^{cu}(\sigma)$ associated is divided by $W^u(\sigma)$ and $W^s(\sigma) \cap W^{cu}(\sigma)$ in the four sectors s_{11} , s_{12} , s_{21} , s_{22} . $\pi : V_\sigma \rightarrow W^{cu}(\sigma)$ is the projection defined in a neighborhood V_σ of σ . Figure 3 exhibits the case when $\pi(M(X) \cap V_\sigma)$ intersects s_{11} and s_{12} .

Lemma 4.3. *Consider σ a Lorenz-like singularity of a Venice mask X , and O a hyperbolic periodic orbit satisfying $Cl(W^u(O)) \cap W^{s,+}(\sigma) \neq \emptyset$ and $Cl(W^u(O)) \cap W^{s,-}(\sigma) \neq \emptyset$. Moreover, $\pi(Cl(W^u(O))) \cap s_{1i} \neq \emptyset$ and $\pi(Cl(W^u(O))) \cap s_{2i} \neq \emptyset$ for some $i \in \{1, 2\}$. If q is a regular point in $W^u(\sigma) \cap Cl(s_{1i}) \cap Cl(s_{2i})$, then $O = \omega_X(q)$.*

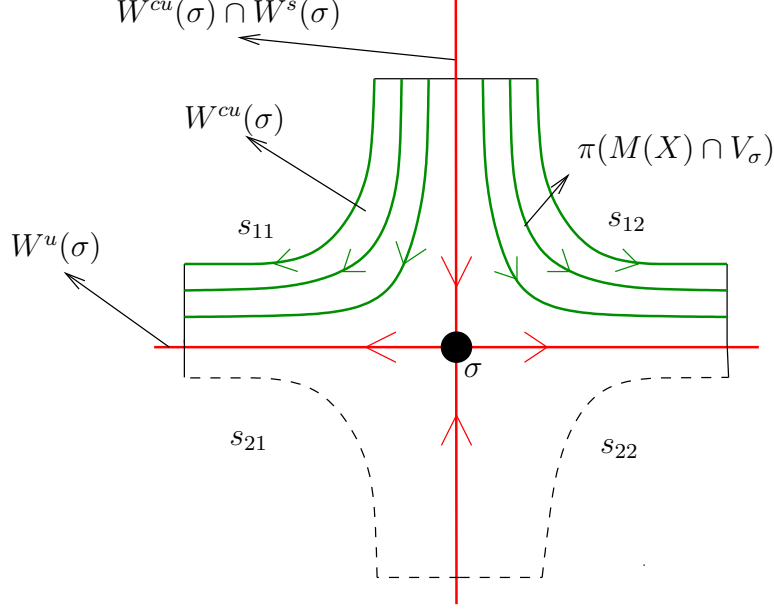


Figure 3: Center unstable manifold of σ .

Proof. We take $q \in W^u(\sigma)$ a regular point close to σ . We assert that $q \in W^s(O)$. Indeed, if we suppose that is not the case, we will get a contradiction.

So, we assume $q \in W^u(\sigma) \setminus W^s(O)$. Then, there is a sequence $p_n^- \rightarrow q$ such that $p_n^- \in W^u(O)$ for all n . In addition, $\{O_X(p_n^-) : n \in \mathbb{N}\}$ accumulates some regular point p^- in $W^{s,-}(\sigma)$ or in $W^{s,+}(\sigma)$. We can suppose the accumulation in some point of $W^{s,-}(\sigma)$. Also, we can take $\{p_n^+ : n \in \mathbb{N}\} \subset W^u(O)$ be a sequence such that $p_n^+ \rightarrow q$. Moreover, $\{O_X(p_n^+) : n \in \mathbb{N}\}$ accumulates σ and some point p^+ in $W^{s,+}(\sigma)$. We have $p_n^+, p_n^- \notin W^u(\sigma)$ for all n . On the other hand, $q \in Cl(W^u(O))$ and the invariance of $W^u(\sigma)$ imply $O_X(q) \subset Cl(W^u(O))$. But $Cl(W^u(O))$ is a closed set, therefore $Cl(O_X(q)) \subset Cl(W^u(O))$. Applying the compactness of $Cl(W^u(O))$ and Tubular Flow Box Theorem [28] in a neighborhood of $O^+(q)$ we obtain that $\{O^+(p_n^+) : n \in \mathbb{N}\}$ and $\{O^+(p_n^-) : n \in \mathbb{N}\}$ accumulate all point in $W^u(\sigma)$ close to $\omega_X(q)$.

As O and $\omega_X(q)$ are invariant closed sets, then they are disjoint and $d(x, \omega_X(q)) > 0$ for all $x \in O$. This implies that there exists $\varepsilon > 0$ such that every point y close to $\omega_X(q)$ satisfies $d(y, O) > \varepsilon$. Moreover $y \notin O_X(q)$ and, $\{O^+(p_n^+) : n \in \mathbb{N}\}$, $\{O^+(p_n^-) : n \in \mathbb{N}\}$ accumulate y . The positive orbits of p_n^+ and p_n^- cannot intersect $\omega_X(q)$. So, we have two possibilities, either any orbit intersects $O_X(q)$, or no orbit does it. The first case means that there is a point

$w \in W^u(\sigma) \cap W^u(O)$ which is absurd. So, neither orbit intersects $O_X(q)$. Now, q is a non-recurrent point. Then, $\{O_X^+(p_n^+) : n \in \mathbb{N}\}$ does not accumulate on $W^{s,+}(\sigma)$. But this contradicts the choice of the sequences. Therefore $q \in W^s(O)$. So, we conclude $O = \omega_X(q)$. \square

From *Lemma 4.3* we obtain the following corollary.

Corollary 4.4. *Consider σ a Lorenz-like singularity of a Venice mask X , and O a hyperbolic periodic orbit satisfying $W^u(O) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(O) \cap W^{s,-}(\sigma) \neq \emptyset$. Let q be a regular point in $W^u(\sigma) \cap Cl(W^u(O))$ and let $\{p_n : n \in \mathbb{N}\} \subset Cl(W^u(O)) \cap W^s(O)$ be a sequence such that $p_n \rightarrow q$. Then $p_n \in O_X(q)$ for all n large.*

Proof. For this is sufficient to observe that $O_X(q)$ is contained in $W^s(O)$. \square

Remark 4.5. *Corollary 4.4 says that for $i \in \{1, 2\}$ and for every hyperbolic periodic orbit O of X , is not possible $H(O) \cap s_{1i} \neq \emptyset$ and $H(O) \cap s_{2i} \neq \emptyset$ simultaneously.*

Lemma 4.6. *Let σ be a singularity of a Venice mask X , and let O be a hyperbolic periodic orbit such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$. Then for $q \in W^u(\sigma) \setminus \{\sigma\}$, $\omega_X(q)$ has singular partitions of arbitrarily small size.*

Proof. We adapt the proof of *Theorem 17* given in [5]. Observe that $\omega_X(q)$ is sectional-hyperbolic. Therefore, if $\omega_X(q)$ is a closed orbit, then *Theorem 3.9* implies that q satisfies the property $(P)_\Sigma$ for some closed subset Σ . Moreover, we can apply *Theorem 16* in [5] to conclude that $\omega_X(q)$ has singular partitions of arbitrarily small size.

Hereafter, we assume $\omega_X(q)$ is not a closed orbit. By *Proposition 3* in [5] is sufficient to prove that for all $z \in \omega_X(q)$ there is cross section Σ_z close to z such that $z \in Int(\Sigma_z)$ and $\omega_X(q) \cap \partial\Sigma_z = \emptyset$.

We assert that $\omega_X(q)$ cannot contain any local strong stable manifold. Indeed, we first assume that $\omega_X(q)$ has no singularities. By *Hyperbolic lemma*, it is hyperbolic saddle-type. Suppose $\omega_X(q)$ containing a local strong stable manifold. Then, by *Lemma 11* in [5], q would be a recurrent point. Therefore using *Lemma 5.6* in [22], there is $x^* \in Per(X) \cap \omega_X(q)$ such that $q \in W_X^s(x^*)$. This means that $\omega_X(q)$ is a periodic orbit which contradicts our assumption. Now, if $\omega_X(q)$ is a sectional-hyperbolic set with singularities, applying *Main*

Theorem in [20], $\omega_X(q)$ cannot contain any local strong stable manifold.

We can fix a foliated rectangle of small diameter R_z^0 such that $z \in \text{Int}(R_z^0)$ and $\omega_X(q) \cap \partial^h R_z^0 = \emptyset$. By *Theorem* 3.1, the intersection of $W^u(O)$ with $W^s(\sigma)$ occurs in some connected component $W^{s,+}(\sigma)$ or $W^{s,-}(\sigma)$ (or both). We initially assume the intersection in $W^{s,+}(\sigma)$.

Since $z \in \omega_X(q)$ and the omega-limit set is not a closed orbit, we have that the positive orbit of q intersects either only one or the two connected components of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$.

Assume the intersection is occurring in just one component only, we shall consider the following cases:

- $W^{s,-}(\sigma) \cap M(X) = \emptyset$.

Using this and linear coordinates around σ , we can construct an open interval $I^+ = I_q^+ \subset W^u(O)$, contained in a suitable cross section through $q \in W^u(\sigma) \setminus \{\sigma\}$ and $q \in \partial I^+$. As $W^u(O) \cap W^{s,+}(\sigma)$ is dense in $W^u(O)$ we have $I^+ \cap W^{s,+}(\sigma)$ is dense in I^+ .

It is possible to assume I^+ is contained in that component of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$. It is because of the positive orbit of q carries the positive orbit of I^+ into such a component. Furthermore, the stable manifolds through I^+ form a subrectangle R_I^+ in there. So, $W^{s,+}(\sigma) \cap R_I^+$ is dense in R_I^+ .

Now, as in *Theorem* 17 of [5], we suppose $\omega_X(q) \cap \text{Int}(R_I^+) \neq \emptyset$ to obtain a contradiction. By hypothesis, the omega-limit set of q is not a periodic orbit. Then *Lemma* 5.6 in [22] implies that the positive orbit of q cannot intersect $\mathcal{F}^s(q, R_z^0)$ infinitely many times. Now, if it intersects R_I^+ , then by the density of $W^{s,+}(\sigma) \cap R_I^+$ in R_I^+ , we can assert that the positive orbit of a point p in $W^{s,-}(\sigma)$ would intersect R_I^+ . Therefore $p \in Cl(W^u(O)) \subset M(X)$ which we get a contradiction. So $\omega_X(q) \cap \text{Int}(R_I^+) = \emptyset$.

To continue, we choose a point $z' \in \text{Int}(R_I^+)$ and a point z'' in the connected component $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$ not intersected by the positive orbit of q . The desired rectangle Σ_z is a subrectangle of R_z^0 bounded by $\mathcal{F}^s(z', R_z^0)$ and $\mathcal{F}^s(z'', R_z^0)$.

- $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ and $W^s(\sigma) \cap W^u(O') \subset W^{s,-}(\sigma)$ for some hyperbolic periodic orbit $O' \neq O$.

In this way, we have the hypotheses of *Theorem 17* in [5]. Therefore there exists an interval $I^- \subset W^u(O')$ contained in that component of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$, such that $q \in \partial I^-$ and $I^- \cap W^{s,-}(\sigma)$ is dense in I^- . The stable manifolds throught $I = I^+ \cup \{q\} \cup I^-$ form a subrectangle R_I in there, with $\text{Int}(R_I) \cap \omega_X(q) = \emptyset$. So, the existence of Σ_z is guaranteed such as last item.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap W^u(O) \neq \emptyset$.

We assert that there are O_1, O_2 hyperbolic periodic orbits such that, $W^s(\sigma) \cap W^u(O_1) \subset W^{s,+}(\sigma)$ and $W^s(\sigma) \cap W^u(O_2) \subset W^{s,-}(\sigma)$. Indeed, we take $q_1 \in W^{s,+}(\sigma) \cap W^u(O)$ and $q_2 \in W^{s,-}(\sigma) \cap W^u(O)$.

As $M(X)$ is union of homoclinic classes and $W^u(O) \subset M(X)$, there are hyperbolic periodic orbits O_1, O_2 satisfying $q, q_1 \in H(O_1)$ and $q, q_2 \in H(O_2)$. Therefore $O_X(q_1) \subset H(O_1)$ and $O_X(q_2) \subset H(O_2)$. Moreover, since the homoclinic classes are closed set we have that σ and O are in $H(O_1) \cap H(O_2)$. From *Remark 4.5* follows $H(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $H(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. On the other hand, let $W^+(O)$ be the connected component of $W^u(O) \setminus O$ containing q_1 , then $W^+(O) \subset H(O_1)$. Analogously, for $W^-(O)$, the connected component of $W^u(O) \setminus O$ containing q_2 , we have $W^-(O) \subset H(O_2)$. Therefore $W^u(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $W^u(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. Again we have the hypotheses of *Theorem 17* in [5].

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap H(O) \neq \emptyset$.

It is not possible by *Corollary 4.4*.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$, $W^{s,-}(\sigma) \cap Cl(W^u(O')) \neq \emptyset$ and $q \in Cl(W^u(O'))$, where O' is a hyperbolic periodic orbit of X .

From last item $O' \notin H(O)$. As X satisfies the Property (P), there is $\sigma' \in \text{Sing}(X)$ such that $W^u(O') \cap W^s(\sigma') \neq \emptyset$. If $\sigma' = \sigma$ then $W^u(O')$ intersects $W^{s,+}(\sigma)$ or $W^{s,-}(\sigma)$. Observe that those alternatives were already analyzed. If $\sigma' \neq \sigma$, then we can obtain an interval J^- such that $J^- \subset W^u(O')$ and $J^- \cap W^s(\sigma')$ is dense in J^- . Moreover we can assume $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ to obtain an interval I^+ such that $I^+ \subset W^u(O)$ and $I^+ \cap W^{s,+}(\sigma)$ is dense in I^+ . Since $O' \notin H(O)$, follows that $W^u(O') \not\subset H(O)$. Therefore $W^u(O')$ cannot intersect $W^{s,+}(\sigma)$. In this way, there is an open arc $I^- \subset \bigcup_{t \geq 0} X_t(J^-)$ such that $q \in \partial I^-$. I^- works such as in second item. The stable manifolds throught $I = I^+ \cup \{q\} \cup I^-$ generates a subrectangle R_I . This acts such as *Theorem 17* in [5].

Now assume the positive orbit intersects both components of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$. Therefore we take I (or I^+ to first case) with the positive orbit as before to obtain two subrectangles R_I^t and R_I^b , like R_I (or R_I^+ to first case), in each component. Then we select two points $z' \in \text{Int}(R_I^t)$ and $z'' \in \text{Int}(R_I^b)$ and define Σ_z as the rectangle in R_z^0 bounded by $\mathcal{F}^s(z', R_z^0)$ and $\mathcal{F}^s(z'', R_z^0)$.

From *Proposition 3* in [5] we conclude the result. \square

We remember the concept of *singular cross section* that appears in [24]. For a disjoint collection of rectangles $\mathcal{S} = \{S_1, \dots, S_l\}$ we denote $\mathcal{S}^o = \mathcal{S} \setminus \partial\mathcal{S}$ and $\partial^*\mathcal{S} = \bigcup_{S \in \mathcal{S}} \partial^*S$ for $*$ = h, v, o .

Definition 4.7. A *singular cross section* of X is a finite disjoint collection \mathcal{S} of foliated rectangles with $M(X) \cap \partial^h\mathcal{S} = \emptyset$ such that for every $S \in \mathcal{S}$ there is a leaf l_S of \mathcal{F}^s in S^o such that the return time $t_S(x)$ for $x \in S \cap \text{Dom}(\Pi_S)$ goes uniformly to infinity as x approaches l_S .

We define the *singular curve* of \mathcal{S} as the union,

$$l_{\mathcal{S}} = \bigcup_{S \in \mathcal{S}} l_S.$$

Proposition 4.8. Let q be a regular point in $W^u(\sigma)$, with σ a singularity of a Venice mask X , and let O be a hyperbolic periodic orbit such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$. Then $\omega_X(q)$ is a closed orbit.

Proof. If $\omega_X(q)$ is a singularity, then it is done. Hereafter, we assume that $\omega_X(q)$ is not a singularity. From *Lemma 4.6* follows that $\omega_X(q)$ has singular partitions of arbitrarily small size. On the other hand, let $T_U M = \hat{F}_U^s \oplus \hat{F}_U^c$ be a continuous extension of the sectional-hyperbolic splitting $T_{\omega_X(q)} M = F_{\omega_X(q)}^s \oplus F_{\omega_X(q)}^c$ of $\omega_X(q)$ to a neighborhood U of $\omega_X(q)$. Let I be an arc tangent to \hat{F}_U^c , transverse to X , with q as boundary point. *Theorem 18* in [5] guarantees for every singular partition $\mathcal{R} = \{S_1, \dots, S_k\}$ of $\omega_X(q)$, the existence of $S \in \mathcal{R}$, $\delta > 0$, a sequence $q'_1, q'_2, \dots \in S$ in the positive orbit of q , and a sequence of intervals $J'_1, J'_2, \dots \subset S$ in the positive orbit of I with q'_j as a boundary point of J'_j for all such that $\text{length}(J'_j) \geq \delta$, for all $j = 1, 2, 3, \dots$.

We can assume $I = J'_1$. As $q, q'_j \in M(X)$ and X is a Venice mask, we can use the *Lemma 3.5* to obtain a sequence $\{q_n : n \in \mathbb{N}\} \subset M$ such that $q_n \rightarrow q$ and $\omega(q_n)$ is a singularity for any n . As X has just a finite singular points, we can take $\omega(q_n) = \{\sigma'\}$ for all n , and some $\sigma' \in \text{Sing}(X)$. If $q_n \in W^u(\sigma)$ for all n , then $\omega(q) = \{\sigma'\}$ which contradicts our assumption. Therefore $q_n \notin W^u(\sigma)$ for any n . We can take q_n such that $q_n \in S$ for all n .

On the other hand, for σ' are possible the following two alternatives, either $\sigma' \in \omega_X(q)$, or $\sigma' \notin \omega_X(q)$. We begin to consider $\sigma' \in \omega_X(q)$. *Lemma 14* in [5] asserts $O^+(q) \cap \mathcal{R} = \{\hat{q}_1, \hat{q}_2, \dots\}$ an infinite sequence ordered in a way that $\Pi(\hat{q}_n) = \hat{q}_{n+1}$, and the existence of a curve $c_n \subset W^s(\text{Sing}(X) \cap \omega_X(q)) \cap B_\delta(\hat{q}_n)$ such that

$$B_\delta^+(\hat{q}_n) \subset \text{Dom}(\Pi) \quad \text{and} \quad \Pi|_{B_\delta^+(\hat{q}_n)} \text{ is } C^1,$$

where $B_\delta^+(\hat{q}_n)$ denotes the connected component of $B_\delta(\hat{q}_n) \setminus c_n$ containing \hat{q}_n . In particular, we can reduce δ to obtain $\Pi_S = \Pi|_S$ such that

$$(\Pi_S)|_{B_\delta^+(q)} \text{ is } C^1.$$

However $W^s(\sigma')$ accumulates q on S , so we obtain a contradiction.

Therefore the first alternative cannot occur. We conclude $\sigma' \notin \omega_X(q)$.

Hartman-Grobman's Theorem implies the existence of a neighborhood $V_{\sigma'}$ of σ' , where the flow is C^0 -conjugated to its linear part. Let $\eta > 0$ be such that $V_{\sigma'} \subset B_\eta(\sigma')$ and $O^+(q) \cap V_{\sigma'} = \emptyset$. From *Lemma 2.2* in [24] there are singular cross sections $\Sigma^+, \Sigma^- \subset V_{\sigma'}$ such that every orbit of $M(X)$ passing close to some point in $W^{s,+}(\sigma')$ (respectively $W^{s,-}(\sigma')$) intersects Σ^+ (respectively Σ^-). Moreover *Lemma 2.3* in [3] guarantees the existence of two disks $\Lambda^+, \Lambda^- \subset V_{\sigma'}$ transverse to X such that for $B_\varepsilon(\sigma') \subset V_{\sigma'}$, and for any point $x \in B_\varepsilon(\sigma')$, there are two numbers $t_- < 0 < t_+$ with $X_{t_-}(x) \in \Sigma^+ \cup \Sigma^-$ and $X_{t_+}(x) \in \Lambda^+ \cup \Lambda^-$. In addition, $X_t(x) \in V_{\sigma'}$ for all $t \in (t_-, t_+)$. See Figure 4.

As $q_n \rightarrow q$, we can take a sequence of open arcs I_1, I_2, \dots with q_n as a boundary point of I_n such that $Cl(I_n)$ converges to $Cl(I)$. In particular, we can assume $\delta \leq \text{length}(I_n) < \epsilon$ for all $n = 1, 2, 3, \dots$ and $\text{diam}(S) = \epsilon$. In addition, we can take $I_n \subset S$ for all n . On the other hand, $q_n \in W^s(\sigma')$ implies that $O^+(q_n)$ intersects $\Sigma^+ \cup \Sigma^-$. Assume that the intersection occurs in Σ^+ for all n . As we can choose the singular partition of arbitrarily small size and q is non-recurrent, there is $\varepsilon' > 0$ such that $\text{diam}(\mathcal{R}) = \varepsilon'$ and $O^+(s_n) \cap \Sigma^+ \neq \emptyset$ for all $s_n \in I_n$.

We assert that q satisfies the property $(P)_\Sigma$, where $\Sigma = \Sigma^+$. Indeed, from $O^+(q) \cap V_{\sigma'} = \emptyset$ follows $O^+(q) \cap \Sigma^+ = \emptyset$. Now, for $x \in I$ there are $\beta_1, \beta_2 > 0$ such that $B_{\beta_1}(x) \cap \partial I = \emptyset$, $B_{\beta_2}(x) \cap \{q_l\} = \emptyset$ and $B_{\beta_2}(x) \cap I_l \neq \emptyset$ for all l large. We define $\beta = \min\{\beta_1, \beta_2\}$. Let $\{x_l\}_l$ be a sequence with $x_l \in I_l \cap B_\beta(x)$ such that $x_l \rightarrow x$. As in [5], we define the *holonomy map* Π_{S, Σ^+} from S to Σ^+ by

$$\text{Dom}(\Pi_{S, \Sigma^+}) = \{y \in S : X_t(y) \in \Sigma^+ \text{ for some } t > 0\}$$

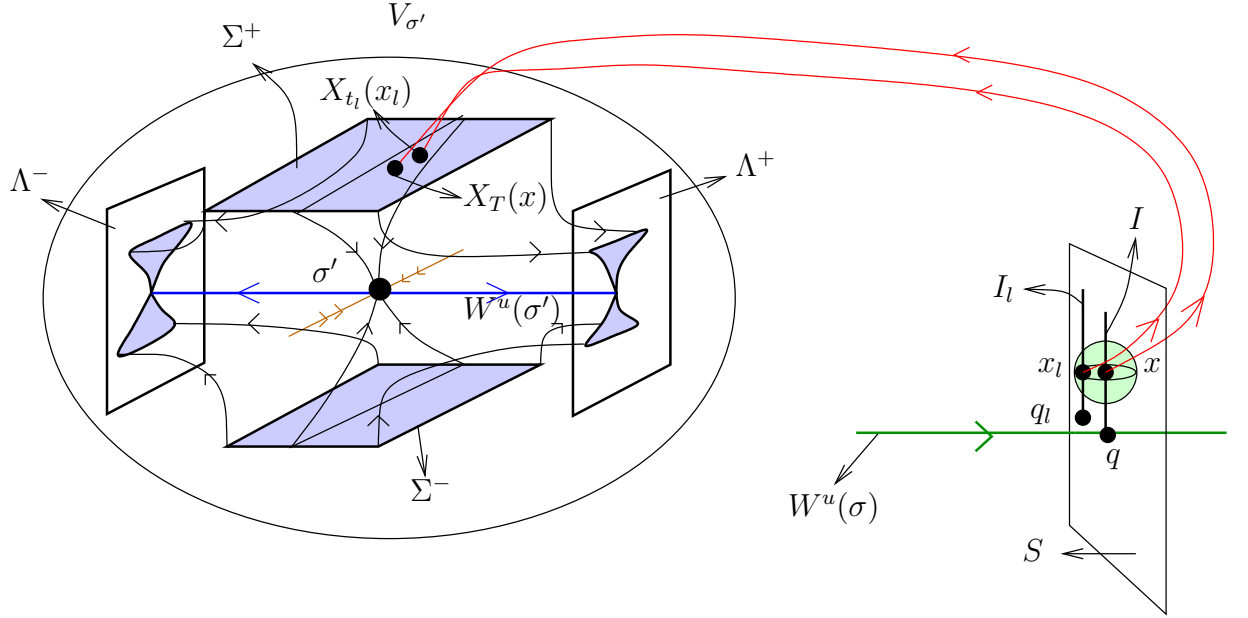


Figure 4: Proof *Proposition 4.8*.

and

$$\Pi_{S,\Sigma^+}(y) = X_{t_{S,\Sigma^+}(y)}(y),$$

where $t_{S,\Sigma^+}(y) = \inf\{t > 0 : X_t(y) \in \Sigma^+\}$.

Therefore $x_l \in \text{Dom}(\Pi_{S,\Sigma^+})$ for all n . From *Lemma 19* and *Theorem 22* in [5] follows that $x \in \text{Dom}(\Pi_{S,\Sigma^+})$.

Finally, *Theorem 3.9* implies that $\omega_X(q)$ is a closed orbit. As we assume $\omega_X(q)$ not being a singularity, then we conclude that the omega-limit set of q is a periodic orbit. □

4.2 Property $(P_{\sigma'})_q^+$

Definition 4.9. Let $\sigma, \sigma' \in \text{Sing}(X)$ and q be a regular point in $W^u(\sigma)$. We say that an open arc $I \subset M$ satisfies the Property $(P_{\sigma'})_q^+$ if $q \in \partial I$ and $I \cap W^{s,+}(\sigma')$ is dense in I . In a similar way, an open arc $J \subset M$ satisfies the Property $(P_{\sigma'})_q^-$ if $q \in \partial J$ and $J \cap W^{s,-}(\sigma')$ is dense in J .

$J_1 \subset \bigcup_{t \geq 0} X_t(J) \subset W^u(O)$ contained in a suitable cross section through w , such that $w \in \partial J_1$. From *Inclination lemma* [28], follows that $W^u(O)$ accumulates points in some branch of $W^u(\sigma_1)$. Therefore, for $q_1 \in (W^u(\sigma_1) \cap Cl(W^u(O))) \setminus \{\sigma_1\}$ there is an open arc I_1 such that $I_1 \subset \bigcup_{t \geq 0} X_t(J_1)$ and $q_1 \in \partial I_1$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^u(O)$ implies the density of $W^{s,+}(\sigma') \cap I_1$ in I_1 . Then I_1 satisfies $(P_{\sigma'})_{q_1}^+$.

- When the omega-limit set of p_1 and p_2 are respectively hyperbolic periodic orbits O_1, O_2 , we have that $W^u(O_i)$ intersects the stable manifold of some singularity σ_i of X , $i = 1, 2$. We first assume $\sigma_1 = \sigma_2 = \sigma'$. That intersection cannot just only occurs in $W^s(\sigma')$ because of this would imply $\sigma \notin Cl(W^u(O_1) \cup W^u(O_2))$ and $Cl(W^u(O)) \subset Cl(W^u(O_1) \cup W^u(O_2))$. But $\sigma \in Cl(W^u(O))$ which produces a contradiction. Therefore we can assume that $W^u(O_1) \cap W^s(\sigma_1) \neq \emptyset$ with $\sigma_1 \neq \sigma'$.

Applying *Inclination lemma*, $Cl(W^u(O))$ and $\bigcup_{t \geq 0} X_t(J)$ intersect $W^s(\sigma_1)$ transversally. Again, let $w \in W^u(O) \cap W^s(\sigma)$ be a point in $\bigcup_{t \geq 0} X_t(J)$ close to σ_1 . Using it and linear coordinates around σ_1 , we can construct an open interval $J_1 \subset W^u(O)$ contained in a suitable cross section through w . $J_1 \setminus \{w\}$ is formed by two open arcs $J_1^+, J_1^- \subset W^u(O)$. Therefore, for $q_1 \in W^u(\sigma_1) \setminus \{\sigma_1\}$ there is an open arc I_1 such that $q_1 \in \partial I_1$ and, $I_1 \subset \bigcup_{t \geq 0} X_t(J^+)$, or $I_1 \subset \bigcup_{t \geq 0} X_t(J^-)$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^{s,+}(\sigma')$ implies the density of $W^{s,+}(\sigma) \cap I_1$ in I_1 . Then I_1 satisfies $(P_{\sigma'})_{q_1}^+$.

If $\sigma_1 = \sigma$, then the result is obtained. Otherwise, we apply a similar process to σ_1 to get $\sigma_3 \in Sing(X)$ with $\sigma_3 \notin \{\sigma', \sigma_1\}$, and an open arc $I_3 \subset Cl(W^u(O))$ such that I_3 satisfies the Property $(P_{\sigma'})_{q_3}^+$.

As $\sigma \in Cl(W^u(O))$ and X just has finitely many singularities, we conclude the existence of some open arc satisfying the Property $(P_{\sigma'})_q^+$ for $q \in W^u(\sigma) \cap Cl(W^u(O))$.

□

4.3 Proof of Theorem A

It is sufficient to prove the existence of singular partitions of arbitrarily small size.

Let q be a regular point in $W^u(\sigma)$, where $\sigma \in Sing(X)$.

As $M(X)$ is union of homoclinic classes, there is a hyperbolic periodic orbit O such that σ and q are contained in the homoclinic class associated to O , denoted by $H(O)$. In addition $H(O)$ intersects only one or the two connected

components $W^{s,+}(\sigma), W^{s,-}(\sigma)$ of $W^s(\sigma) \setminus \mathcal{F}_X^{ss}(\sigma)$. We begin to analyze the intersection in $W^{s,+}(\sigma)$. On the other hand, X satisfies the Property (P). This implies that there is a singularity $\sigma' \in \text{Sing}(X)$ with $W^u(O) \cap W^s(\sigma') \neq \emptyset$. By *Theorem 3.1*, the intersection of $W^u(O)$ with $W^s(\sigma')$ is either only one or the two connected components $W^{s,+}(\sigma'), W^{s,-}(\sigma')$ of $W^s(\sigma') \setminus \mathcal{F}_X^{ss}(\sigma')$. If $\sigma = \sigma'$ then from *Lemma 4.6* follows the existence of singular partitions of arbitrarily small size. Hereafter, we assume $\sigma \neq \sigma'$ and $W^{s,+}(\sigma') \cap W^u(O) \neq \emptyset$.

If $Cl(W^u(O)) \cap W^{s,-}(\sigma') \neq \emptyset$, then *Lemma 4.3* and *Proposition 4.8* imply that for some $p \in W^u(\sigma') \cap Cl(W^u(O))$, $O = \omega_X(p)$ and $H(O) \subset Cl(W^u(\sigma'))$. But $q \notin W^u(\sigma')$. This contradicts $q \in H(O)$. So, $Cl(W^u(O)) \cap W^{s,-}(\sigma') = \emptyset$. *Proposition 4.10* guarantees the existence of an open arc $I^+ \subset M$ satisfying the Property $(P_{\sigma'})_q^+$.

We suppose $\omega_X(q)$ is not a periodic orbit. Let z be a point in $\omega_X(q)$. In a similar way as *Lemma 4.6*, we fix a foliated rectangle of small diameter R_z^0 such that $z \in \text{Int}(R_z^0)$ and $\omega_X(q) \cap \partial^h R_z^0 = \emptyset$. The positive orbit of q intersects either only one or the two connected components of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$.

Assume the intersection is occurring in just one component only.

Now, analyze the following cases:

- $q \notin H(O')$ for all hyperbolic periodic orbit O' of X such that $H(O') \cap W^{s,-}(\sigma) \neq \emptyset$.

The existence of the singular partitions of arbitrarily small size is obtained such as the first case in *Lemma 4.6*.

- There is a sequence $\{p_n\}_n \subset W^u(O)$ such that $p_n \rightarrow p \in W^{s,-}(\sigma)$, and there is a sequence $\{q_n\}$ such that $q_n \in O_X(p_n)$ and $q_n \rightarrow q$.

From *Lemma 4.3* follows that $\omega_X(q) = O$. But this contradicts our assumption that the omega-limit set is not a periodic orbit.

- For some periodic orbit $O' \neq O$, there is a sequence $\{p_n : n \in \mathbb{N}\} \subset W^u(O')$ such that $p_n \rightarrow p \in W^{s,-}(\sigma)$, and there is a sequence $\{q_n : n \in \mathbb{N}\}$ satisfying $q_n \in O_X(p_n)$ and $q_n \rightarrow q$.

Again, *Lemma 4.3* implies that $W^u(O')$ does not intersect the open arc I^+ . From Property (P), there is $\sigma'' \in \text{Sing}(X)$ such that $W^u(O') \cap W^s(\sigma'') \neq \emptyset$. Then for some $r \in W^u(\sigma'')$ there is an interval $J^- \subset W^u(O')$, such that $r \in \partial J$ and $J^- \cap W^s(\sigma'')$ is dense in J^- . Also there is an open arc $I^- \subset \bigcup_{t \geq 0} X_t(J^-)$ satisfying $q \in \partial I^-$. Therefore $I^- \subset W^u(O')$ and $I^- \cap W^s(\sigma'')$ is dense in I^- . In addition, $W^{s,+}(\sigma) \cap I^- = \emptyset$. The stable manifolds through

$I = I^+ \cup \{q\} \cup I^-$ generates a subrectangle R_I . This rectangle acts such as *Lemma 17* in [5].

The existence of the singular partition of arbitrarily small size is obtain such as *Lemma 4.6*.

If the intersection of $O_X^+(q)$ with R_z^0 occurs in both connected components of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$, then we proceed such as *Lemma 4.6* to get a cross section Σ_z with $z \in \Sigma_z$ and $\partial\Sigma_z \cap \omega_X(q) = \emptyset$.

In this way, *Proposition 3* in [5] implies the existence of the singular partition of arbitrarily small size for $\omega_X(q)$.

Finally, we follow the proof of *Proposition 4.8* to conclude that $\omega_X(q)$ is a closed orbit.

5 Intersection of homoclinic classes

In this section we are interested in the study of the intersection of homoclinic classes in a sectional-Anosov flow. We follow some ideas developed in [8] to obtain *Theorem B*. More specifically, we prove that in this context, this intersection can be decomposed in three specific sets: a non-singular hyperbolic set, finitely many singularities and regular orbits joining them. Recall that an invariant set is nontrivial if it does not reduce to a single orbit. The conclusion of *Theorem B* is obvious when H_1 or H_2 are trivial invariant sets. Hereafter, H_1 and H_2 are two non trivial different homoclinic classes in $M(X)$. Let Λ be the intersection between H_1 and H_2 . We start with the following lemma.

Lemma 5.1. *Assume that there is a singularity $\sigma \in \Lambda$, then for $\delta > 0$ small, every sequence $\{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma)$ such that $x_n \rightarrow \sigma$ is contained in $W^s(\sigma) \cup W^u(\sigma)$.*

Proof. We suppose by contradiction that there is a sequence $\{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma)$ such that $x_n \rightarrow \sigma$ and $x_n \notin W^s(\sigma) \cup W^u(\sigma)$ for all n .

So, we obtain two sequences x_n^s and x_n^u , in the orbit of x_n such that $x_n^s \rightarrow y^s$ and $x_n^u \rightarrow y^u$ for some $y^s \in W^s(\sigma) \setminus \{\sigma\}$ and $y^u \in W^u(\sigma) \setminus \{\sigma\}$ close to σ . Let O_1, O_2 be two orbits such that $H(O_1) = H_1$ and $H(O_2) = H_2$. Then there exist sequences $\{p_n : n \in \mathbb{N}\} \subset (W^u(O_1) \cap W^s(O_1))$ and $\{q_n : n \in \mathbb{N}\} \subset (W^u(O_2) \cap W^s(O_2))$ satisfying $p_n \rightarrow x_n^s$ and $q_n \rightarrow x_n^u$. We can assume $p_n \notin H_2$ for all n . This means that $p_n \rightarrow x^s$ and $q_n \rightarrow x^s$ too. The behavior of the orbits of x_n, p_n and q_n nearby σ , are as described in Figure 6.

Since homoclinic classes have density of periodic points [16], for each n we have that p_n and q_n are approximated respectively by a sequence of periodic

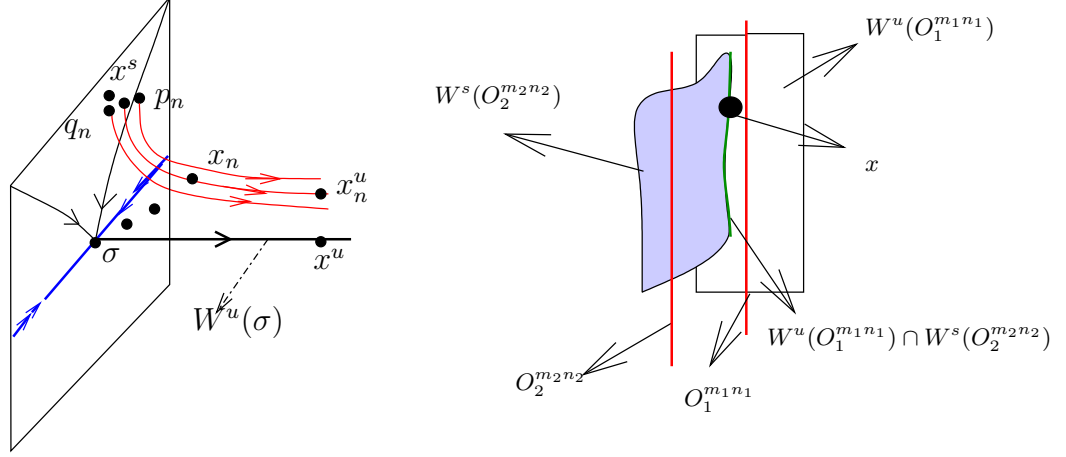


Figure 6: *Lemma 5.1*

orbits $\{O_1^{mn} : m \in \mathbb{N}\}$ and $\{O_2^{mn} : m \in \mathbb{N}\}$. Define the map $\pi : B_\delta(\sigma) \rightarrow W^{cu}(\sigma)$ such as in *Subsection 4.1*. Observe that $\{\pi(W^u(O_1^{mn})) : m \in \mathbb{N}\}$ and $\{\pi(W^u(O_2^{mn})) : m \in \mathbb{N}\}$ accumulate y^s in the same sector s_{ij} of $W^{cu}(\sigma)$. Follows from *Lemma 3.1* in [12] that these sequences can be chosen in a way that, for $i = 1, 2$ and for all n, m , $W^s(O_i^{nm})$ is uniformly bounded away from zero. This implies that for m_1, m_2, n_1, n_2 large, $W^u(O_1^{m_1 n_1}) \cap W^s(O_2^{m_2 n_2}) \neq \emptyset$. Consider $x \in W^u(O_1^{m_1 n_1}) \cap W^s(O_2^{m_2 n_2})$. As $O_1^{m_1 n_1} \subset (H_1 \setminus H_2)$ and $O_2^{m_2 n_2} \subset H_2$, then there is $x^* \in O_X(x)$ such that $x^* \in \Lambda$. But Λ is an invariant closed set, then $O_1^{m_1 n_1} \subset Cl(O_X(x^*)) = Cl(O_X(x^*)) \subset \Lambda$. However $O_1^{m_1 n_1} \not\subset H_2$ and $\Lambda \subset H_2$, which is a contradiction.

We conclude $x_n \in W^s(\sigma) \cup W^u(\sigma)$ for all $n \in \mathbb{N}$.

□

5.1 Proof theorem B

Theorem B gives a description about the set Λ .

Proof. The idea of the proof is the same given in *Lemma 3.3* by [8]. Follows to *Lemma 5.1* that there is $\delta > 0$ such that $\Lambda \cap B_\delta(\sigma) \subset W^s(\sigma) \cup W^u(\sigma)$, and the balls $B_\delta(\sigma)$ are pairwise disjoint for every $\sigma \in \Lambda \cap \text{Sing}(X) = S$. Define

$$H = \bigcap_{(t, \sigma) \in \mathbb{R} \times S} X_t(\Lambda \setminus B_\delta(\sigma)).$$

By construction, H is a non-singular, compact invariant sectional-hyperbolic set. So, applying *Lemma 3.2* we have that H is hyperbolic. Now define $R =$

$\Lambda \setminus (S \cup H)$. For $x \in R$ there is $(t, \sigma) \in \mathbb{R} \times S$ with $X_t(x) \in B_\delta(\sigma)$, and by *Lemma 5.1* $X_t(x) \in W^s(\sigma) \cup W^u(\sigma)$.

If $x \in W^u(\sigma)$ we obtain $\alpha(x) \subset H \cup S$. Assume $X_s(x) \notin \bigcup_{\rho \in S} B_\delta(\rho)$ for all $s \geq 0$, then $\omega(x) \subset H$. Now, if there is $(s, \rho) \in \mathbb{R} \times S$ such that $X_s(x) \in B_\delta(\rho)$ then $x \in W^s(\rho)$, So $\omega(x) \in H \cup S$.

With a similar argument we have $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ for $x \in W^s(\sigma)$. So, we conclude the result. \square

6 Some conjectures

Because of the study developed in this work, different questions have appeared. All known examples of Venice mask are characterized because the maximal invariant set is the finite union of homoclinic classes and the intersection between two different homoclinic classes H_1 and H_2 is contained in $Cl(W^u(Sing(X)))$. Moreover, every regular point $q \in W^u(Sing(X)) \cap H_1 \cap H_2$ is non-recurrent.

Consider a Venice mask X supported on a compact 3-manifold M . Let H_1 and H_2 be two different homoclinic classes in $M(X)$ and let Λ be the intersection between H_1 and H_2 . Assume the decomposition of Λ given in *Theorem B*, it is $\Lambda = S \cup H \cup R$.

We announce the following conjecture.

Conjecture 6.1. *Every regular point $q \in R$ is non-recurrent.*

From *Lemma 5.1* we have that for $\delta > 0$ small, $x \in B_\delta(\sigma)$ implies $x \in W^s(\sigma) \cup W^u(\sigma)$ for some $\sigma \in S$. If $x \in W^u(\sigma)$ then $\alpha(x) = \{\sigma\}$. Now we take $x \in W^s(\sigma) \setminus W^u(\sigma)$. Therefore we shall consider two cases, either $\alpha(x) = \{\rho\}$ for some $\rho \in S$ or $\alpha(x) \subset H$. In the first case, we obtain the desired result. If we prove that the second case cannot occur, then the following conjecture would be true.

Conjecture 6.2. $\Lambda \subset Cl(W^u(Sing(X)))$.

Let us state direct consequence of the hyperbolic Lemma 3.2 that appears in [5].

Corollary 6.3. *Every periodic orbit of a sectional-Anosov flow on a compact manifold is hyperbolic. In particular, all such flows have countably many closed orbits.*

This implies that the maximal invariant set of every Venice mask is union of countably many homoclinic classes. So, if *Conjecture 6.1* and *Conjecture 6.2* are true, then would be possible to realize the following statement.

Conjecture 6.4. *The maximal invariant set of every Venice mask is finite union of homoclinic classes.*

Proof. Let X be a Venice mask supported on a compact 3-manifold M . Then X has finite many singularities, we say n . Let H_1, H_2 be two different homoclinic classes associated to $M(X)$. From Conjectures 6.1 and 6.2 is possible to apply Theorem A to conclude that for each singularity σ of X , $Cl(W^u(\sigma)) = \{\sigma\} \cup W^u(\sigma) \cup C_\sigma$, it is a disjoint union and C_σ is a closed orbit. On the other hand, the branches of $W^u(\sigma)$ are uni-dimensional. Therefore Theorem 6.2 implies $H_1 \cap H_2$ has just only a finite number of possibilities to occur. Moreover, at most three homoclinic classes can contain the branch of the unstable manifold of some singularity.

This finishes the proof. □

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